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# Renormalization of the nonequilibrium dynamics of fermions in a flat FRW universe.

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**Abstract**

We derive the renormalized equations of motion and the renormalized energy-momentum tensor for fermions coupled to a spatially homogeneous scalar field ( inflaton) in a flat FRW geometry. The fermion back reaction to the metric and to the inflaton field is formulated in one-loop approximation. Having determined the infinite counter terms in an $\overline{MS}$ scheme we formulate the finite terms in a form suitable for numerical computation. We comment on the trace anomaly which is inferred from the standard analysis. We also address the problem of initial singularities and determine the Bogoliubov transformation by which they are removed.

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# 1 Introduction

The production of fermions during inflation [1] was for a long time considered to be less interesting than the one of scalar quanta. The production of scalars coupled to the inflaton field is characterized, during preheating[2, 3], by parametric resonance bands [2, 4], a feature that leads to an abundant production of such quanta. Numerous authors have investigated various aspects of the production of scalar quanta, including the back reaction to the inflaton field and to the scale parameter [5 − 14]. Also, parametric resonance plays an important rôle in the analysis of the late time behavior of these coupled systems [16, 17], in the large-$N$ limit. For fermions a true parametric resonance cannot develop, as the fermion number is limited by the Pauli principle. So fermions were not considered in the inflationary or preheating stage of inflation but thought to be produced afterwards via the decay of scalar fields. Recently it was observed [18, 19], that the production of fermions during the preheating stage, via their coupling to the inflaton, is characterized by resonance-type bands within which the occupation number comes near saturation. On the other hand, the production of fermions via decay of a boson into an fermion-antifermion pair leads to a sharp momentum spectrum. As a consequence the coherent production of fermions in the inflaton background may largely exceed the production rate obtained via the decay of scalar fields.

This phenomenon has lead several groups to (re)consider the production of fermionic degrees of freedom during the inflationary stage. The nonperturbative production of fermions in an axial background field was considered in [20]. Very massive fermions may produce characteristic spikes in the spectrum of primordial perturbations, leading to observable features in the measured CMB anisotropy and in the power spectrum [21]. In supergravity gravitinos will necessarily be a part of the theory, their parameters can be related to the scale of supersymmetry breaking. The coherent, non thermal production of gravitinos has been considered in Refs. [22, 23, 24]. It may lead to an exceedingly low reheating temperature and then becomes a problem for many models of the early universe [25, 26]. This issue still is still open and being considered by several authors. One of the problems is the modification of the parametric resonance bands by the evolution of the scale parameter, or by an evolution of the inflaton field modified by the back reaction.

Here we will consider the renormalization of the fermionic back-reaction

to the inflaton field and to the energy momentum tensor in an FRW geometry, in one-loop approximation. Though this seems a more formal aspect, it is important for a reliable numerical computation of these back-reaction effects. As long as one just considers the number of produced particles this is not necessary as the adiabatic particle number is finite per se. For the computation of the other quantities simple normal ordering is not sufficient and leaves at least logarithmic divergences.

The renormalization of nonequilibrium dynamics of fermions in Minkowski space has been considered by us previously [18], but this analysis has to be adapted to the FRW geometry. In particular dimensional regularization is modified by the time dependent scale factor. While the flat space counter terms still renormalize the divergent loops the finite terms are different. Furthermore, the structure of the energy momentum tensor in the FRW geometry is modified as compared to the one in Minkowski space. So several new renormalization constants and finite parts have to be determined. For the case of scalar quantum fluctuations we have already presented [9, 10] our scheme [11] for renormalizing the equation of motion for the inflaton and the Friedmann equations. The methods used there, well suited for numerical computations, can be adapted to the case fermions. In this sense the present publication merges our previous work on fermion systems in Minkowski, and scalar fields in FRW space. Another viable scheme is the well-known adiabatic subtraction [27]. While this has been worked out and numerically implemented for the cosmological nonequilibrium evolution of scalar fields in Ref. [8, 15], an application to fermions has not yet been adapted for the requirements of numerical computation.

Of course not all the results presented in this publication are new. The renormalization of the energy momentum tensor of free fermions in a FRW background has been considered long ago, see e.g. [28, 29, 30]; other regularization schemes have been discussed, e.g., in Refs. [31, 32]; the one-loop renormalization of a Yukawa theory is of course standard. However, while the structure of the renormalization constants is known, we have to rederive them here within the nonequilibrium formalism. An important point is their independence of the initial conditions. Furthermore, the remaining finite parts have to be written in a form suitable for numerical computation. As usual, the formulation of the finite parts is the most cumbersome part of renormalization, and this has not been discussed before.

The plan of the paper is as follows: in section 2 we recall the basic

formulae for a flat Friedmann universe, with $3+\epsilon$ space dimensions. In section 3 we embed into this geometry the quantum field theory describing massive fermions, Yukawa-coupled to the scalar inflaton field. The energy momentum tensor for the classical field and the fermionic quantum fluctuations is given in section 4. In section 5 we determine the renormalization constants, and thereby the precise formulae for the finite parts. We also include the trace anomaly that cannot be derived from considering the FRW metric alone. We briefly digress on the question of initial conditions in section 6, conclusions are presented in section 7.

# 2 FRW cosmology

We consider the Friedmann-Robertson-Walker metric with curvature parameter $k = 0$, i. e. a spatially isotropic and flat space-time. The line-element is given in this case by

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 \; . \tag{2.1}$$

The time evolution of the cosmic scale factor $a(t)$ is governed by Einstein's field equation

$$(1+\delta Z)G_{\mu\nu} + \delta\alpha\, {}^{(1)}H_{\;\mu\nu} + \delta\beta\, {}^{(2)}H_{\;\mu\nu} + \delta\gamma\, H_{\;\mu\nu} + \delta\Lambda g_{\;\mu\nu} = -\kappa\langle T_{\mu\nu}\rangle \tag{2.2}$$

with $\kappa = 8\pi G$.

The Einstein curvature tensor $G_{\mu\nu}$ is given by

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \; . \tag{2.3}$$

The Ricci tensor and the Ricci scalar are defined as

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} \; , \tag{2.4}$$

$$R = g^{\mu\nu}R_{\mu\nu} \; , \tag{2.5}$$

where

$$R^{\lambda}_{\alpha\beta\gamma} = \partial_\gamma\Gamma^{\lambda}_{\alpha\beta} - \partial_\alpha\Gamma^{\lambda}_{\gamma\beta} + \Gamma^{\lambda}_{\gamma\sigma}\Gamma^{\sigma}_{\alpha\beta} - \Gamma^{\lambda}_{\alpha\sigma}\Gamma^{\sigma}_{\gamma\beta} \; . \tag{2.6}$$

In FRW geometry with $n-1$ space dimensions, as needed for dimensional regularization, the relevant quantities are given by

$$\begin{aligned}
R_{tt} &= (n-1)\frac{\ddot{a}}{a} \\
R^{\mu}_{\mu} = R &= 2(n-1)\frac{\ddot{a}}{a} + (n-1)(n-2)\left(\frac{\dot{a}}{a}\right)^2 \ .
\end{aligned} \tag{2.7}$$

It is customary to introduce the Hubble parameter $\dot{a}/a$ and to use $R = (n-1)(2\dot{H} + nH^2)$. The tensors ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$, and $H_{\mu\nu}$ arise from the variation of terms proportional to $R^2$, $R^{\alpha\beta}R_{\alpha\beta}$, and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ in the Hilbert-Einstein action. Their general definition is given, e.g., in [30]. Though the fermion loop integrals will not produce divergences proportional to these tensors we will need them as they produce the trace anomaly. The explicit expressions in FRW geometry with $n-1$ space dimensions (see, e.g. [10]) are presented in Appendix A.

The Friedmann equations are obtained by inserting the explicit expressions for the geometrical tensors and for the energy-momentum tensor into the Einstein equations. As this is done after renormalization we can set $n = 4$. Then

$$3H^2 = \kappa\mathcal{E}_{\rm ren} \tag{2.8}$$

$$R = \kappa\mathcal{T}_{\rm ren} \ . \tag{2.9}$$

Here $\mathcal{E} = T_{tt}$ and $\mathcal{T} = T^{\mu}_{\mu}$. For the numerical evolution it is sufficient to consider just the first equation. Then the second one is fulfilled as a consequence of covariant energy conservation

$$\dot{\mathcal{E}} + H(4\mathcal{E} - \mathcal{T}) = 0 \ . \tag{2.10}$$

# 3 Lagrangian and equations of motion

We consider the production of fermions by a scalar field $\Phi$ in a Heisenberg state in which the scalar field has a time-dependent, but spatially homogeneous expectation value

$$\phi(t) = \langle\Phi(t)\rangle \tag{3.1}$$

The Lagrangian density of the model in Minkowski space is given by

$$\begin{aligned}\mathcal{L} &= \frac{1+\delta Z}{2}\partial_\mu\Phi\partial^\mu\Phi - \mathcal{V}(\Phi) - \frac{\xi+\delta\xi}{2}R\Phi^2 - (\zeta+\delta\zeta)R\Phi - \\ &\quad + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_f - g\Phi\right)\psi \end{aligned}\tag{3.2}$$

where

$$\begin{aligned}\mathcal{V}(\Phi) &= (\sigma+\delta\sigma)\Phi + \frac{M^2+\delta M^2}{2}\Phi^2 \\ &\quad + \frac{\kappa+\delta\kappa}{6}\Phi^3 + \frac{\lambda+\delta\lambda}{24}\Phi^4 \end{aligned}\tag{3.3}$$

In addition to the scalar mass $M$, the quartic self-coupling $\lambda$ and the conformal coupling $\xi$ we have introduced a tadpole coupling $\sigma$ , a trilinear coupling $\kappa$ , and an additional coupling to the curvature tensor $\zeta R\Phi$. For $m_f \neq 0$ such terms are generated by the fermion loop and at least the counterterms are needed for the purpose of renormalization. Alternatively one may generate the fermion mass by a shift in $\Phi$, starting with a massless fermion; then the divergent terms with odd powers of $\Phi$ are generated from the terms with even powers of $\Phi$ obtained in the massless case. We have written the counter terms explicitly from the outset though their form is obvious. For the wave function renormalization this is necessary: if we use conformal time and conformal rescaling of the fields in $n = 4 - \epsilon$ dimensions, then part of the kinetic term, including the divergent wave function renormalization, reappears as part of the conformal coupling (see below).

The action in curved space is obtained in the usual way by introducing covariant derivatives and the covariant measure, we have anticipated already the couplings to the curvature tensor. We will implement dimensional regularization in the way of using $n-1$ space components, so we will have to use FRW geometry with $n-1 = 3-\epsilon$ space components, as well.

The equation of motion of the condensate is given by

$$\begin{aligned}(1+\delta Z)\left[\ddot{\phi} + (n-1)\frac{\dot{a}}{a}\dot{\phi}\right] + \frac{d\mathcal{V}(\phi)}{d\phi} + (\zeta+\delta\zeta)R + (\xi+\delta\xi)R\phi(t) \qquad (3.4)\\ + \frac{\lambda+\delta\lambda}{2}\langle\eta^2\rangle + g\langle\overline{\psi}\psi\rangle = 0 \,.\end{aligned}$$

where the back reaction of the fermion field is given, in 1-loop and Hartree approximation by the term $g\langle\overline{\psi}\psi\rangle$, the back-reaction of the scalar field may

be included in one-loop, Hartree, or large-$N$ approximation. The renormalization of the back-reaction of the scalar fluctuations has been discussed recently within our scheme in Refs. [9, 10], and in the adiabatic subtraction scheme in Ref. [8, 15]. So we do not consider these fluctuations here.

It is convenient to introduce conformal time

$$dt = a(\tau)d\tau \tag{3.5}$$

and to rescale the field $\phi$ via

$$\phi(\tau) = a^{-1+\epsilon/2}(\tau)\tilde{\phi}(\tau) \; . \tag{3.6}$$

The derivative with respect to conformal time will be denoted

$$f'(\tau) \equiv \frac{\partial}{\partial\tau}f(\tau) = \partial_\tau f(\tau) \; . \tag{3.7}$$

The classical equation of motion now takes the form

$$\begin{aligned}
&(1+\delta Z)\tilde{\phi}'' + (\zeta+\delta\zeta)a^{3-\epsilon/2}R + a^2\left[\xi+\delta\xi-(1+\delta Z)\frac{n-2}{4(n-1)}R\right]\tilde{\phi} \\
&a^{4-\epsilon}\frac{d\mathcal{V}_\epsilon(\tilde{\phi})}{d\tilde{\phi}} + \frac{\lambda+\delta\lambda}{2}a^\epsilon\tilde{\phi}\langle\eta^2\rangle + ga^{3-\epsilon/2}\langle\overline{\psi}\psi\rangle = 0 \; ,
\end{aligned} \tag{3.8}$$

where we have introduced the rescaled potential

$$\begin{aligned}
\mathcal{V}_\epsilon(\tilde{\phi}) &= (\sigma+\delta\sigma)a^{-1+\epsilon/2}\tilde{\phi} + \frac{M^2+\delta M^2}{2}a^{-2+\epsilon}\tilde{\phi}^2 \\
&\quad + \frac{\kappa+\delta\kappa}{6}a^{-3+3\epsilon/2}\tilde{\phi}^3 + \frac{\lambda+\delta\lambda}{24}a^{-4+2\epsilon}\tilde{\phi}^4
\end{aligned} \tag{3.9}$$

The covariant derivative in the Dirac equation is obtained using the $n$-bein formalism (see Appendix B). One obtains

$$i\gamma^0\left(\partial_0 + \frac{n-1}{2}\frac{\dot{a}}{a}\right)\psi + \left(\frac{i}{a}\gamma^k\nabla_k - m_f - ga^{\epsilon/2}\tilde{\phi}\right)\psi = 0 \; . \tag{3.10}$$

The extra term in the first parenthesis can be eliminated by defining

$$\psi = a^{-(n-1)/2}\tilde{\psi} \; ; \tag{3.11}$$

furthermore, we introduce conformal time and obtain

$$\left[i\partial_\tau - \tilde{\mathcal{H}}(\tau)\right]\tilde{\psi}(\tau,\mathbf{x}) = 0\ . \tag{3.12}$$

with

$$\tilde{\mathcal{H}}(\tau) = -i\boldsymbol{\alpha}\nabla + \left[a(\tau)m_f + g a^{\epsilon/2}(\tau)\tilde{\phi}(\tau)\right]\beta\ . \tag{3.13}$$

Note that $\tilde{\mathcal{H}}$ is rescaled with respect to the Hamiltonian in Minkowski space with a factor $a(\tau)$. So the Dirac equation takes its conventional form, with an effective time-dependent mass

$$m(\tau) = a(\tau)m_f + g a^{\epsilon/2}(\tau)\tilde{\phi}(\tau)\ . \tag{3.14}$$

On account of the spatial homogeneity of the condensate field it is suitable to expand the Dirac field as

$$\tilde{\psi}(\tau,\mathbf{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_0}\left[b_{\mathbf{p},s}U_{\mathbf{p},s}(\tau) + d^\dagger_{-\mathbf{p},s}V_{-\mathbf{p},s}(\tau)\right]e^{+i\mathbf{p}\cdot\mathbf{x}}\ , \tag{3.15}$$

with the time independent creation and annhiliation operators for quanta, whose mass $m_0 = m(0)$ is determined by the initial state. $E_0$ is the corresponding energy $\sqrt{m_0^2+\mathbf{p}^2}$. The creation and annihilation operators satisfy the standard anti-commutation relations

$$\begin{aligned} \{b_{\mathbf{p},s}, b^\dagger_{\mathbf{p}',s'}\} &= 2E_0(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{ss'}\ , &(3.16)\\ \{d_{\mathbf{p},s}, d^\dagger_{\mathbf{p}',s'}\} &= 2E_0(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{ss'}\ . &(3.17)\end{aligned}$$

For the positive and negative energy solutions we make the usual ansatz

$$U_{\mathbf{p},s}(\tau) = N_0\left[i\partial_\tau + \tilde{\mathcal{H}}_{\mathbf{p}}(\tau)\right]f_p(\tau)\begin{pmatrix}\chi_s\\0\end{pmatrix} \tag{3.18}$$

and

$$V_{\mathbf{p},s}(\tau) = N_0\left[i\partial_\tau + \tilde{\mathcal{H}}_{-\mathbf{p}}(\tau)\right]g_p(\tau)\begin{pmatrix}0\\\chi_s\end{pmatrix}\ , \tag{3.19}$$

with the Fourier-transformed Hamiltonian

$$\tilde{\mathcal{H}}_{\mathbf{p}}(\tau) = \boldsymbol{\alpha}\boldsymbol{p} + m(t)\beta\ . \tag{3.20}$$

For the two-spinors $\chi_s$ we use helicity eigenstates, i.e.,

$$\hat{\boldsymbol{p}}\boldsymbol{\sigma}\chi_\pm = \pm\chi_\pm \ . \tag{3.21}$$

The mode functions $f_p$ and $g_p$ depend only on $p = |\mathbf{p}|$; they obey the second order differential equations

$$\left[\frac{d^2}{d\tau^2} - im'(\tau) + p^2 + m^2(\tau)\right] f_p(\tau) \ = \ 0 \ , \tag{3.22}$$

$$\left[\frac{d^2}{d\tau^2} + im'(\tau) + p^2 + m^2(\tau)\right] g_p(\tau) \ = \ 0 \ . \tag{3.23}$$

If the fermion Fock space is based on the conformal vacuum state the modes start - in conformal time - as if the mass was independent of $\tau$ for $\tau \leq 0$. The mode functions, which would be plane waves for $t \leq 0$, then satisfy the initial conditions

$$f_p(0) = 1 \quad , \quad \dot{f}_p(0) = -iE_0 \ , \tag{3.24}$$

$$g_p(0) = 1 \quad , \quad \dot{g}_p(0) = iE_0 \ , \tag{3.25}$$

so that $g(\tau) = f^*(\tau)$. The fermion condensate occurs in the equation of motion as $ga^{3-\epsilon/2}\langle\bar{\psi}\psi\rangle$, the rescaling of the Dirac field changes this to $ga^{\epsilon/2}\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle$. We define a fluctuation integral as

$$\begin{aligned} \mathcal{F}(\tau) &= \langle\bar{\tilde{\psi}}\tilde{\psi}\rangle = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_0} \overline{V}_{-\mathbf{p},s}(t) V_{-\mathbf{p},s}(t) \\ &= -2\int \frac{d^{n-1}p}{(2\pi)^{n-1}2E_0}\left\{2E_0 - \frac{2\mathbf{p}^2}{E_0+m_0}|f_p|^2\right\} \ . \end{aligned} \tag{3.26}$$

We have written the mode integral in dimensionally regulated form. This corresponds to the usual prescription of introducing dimensional regularization 'after taking the Dirac traces'. Indeed the mode integral is a trace of a Green function.

# 4 The energy-momentum tensor

The energy-momentum tensor for a spatially isotropic background field and of the quantum fluctuations generated by this field is diagonal and of the form

$$T^{\mu\nu} = \mathrm{diag}\,(\mathcal{E}, \mathcal{P}, \mathcal{P}, \mathcal{P}) \ . \tag{4.1}$$

From the Lagrangian (3.2) we derive for the energy density of the condensate or background field

$$\begin{aligned}
\mathcal{E}_{\rm cond} &= \frac{1+\delta Z}{2} a^{-4+\epsilon}(\tilde\phi')^2 + \mathcal{V}_\epsilon(\tilde\phi) \\
&+2(n-1)\left[\xi+\delta\xi-\frac{(n-2)(1+\delta Z)}{4n-4}\right] a^{-2+\epsilon}\left(Ha^{-1}\tilde\phi\tilde\phi' - \frac{n-2}{4}H^2\tilde\phi^2\right) \\
&+2(\zeta+\delta\zeta)(n-1)a^{-2+\epsilon/2}H\tilde\phi' + \frac{\delta Z_G}{\kappa}G_{tt} + \frac{\delta\Lambda}{\kappa}\ ,
\end{aligned} \tag{4.2}$$

where $\mathcal{V}_\epsilon(\tilde\phi)$ has been defined above, in Eq. (3.9).

We have included the counter terms proportional to $G_{tt}$ and $g_{tt}$ which appear on the left hand side of the Einstein field equations. They are needed, as the other counter terms, in order to compensate the divergences arising in the fluctuation part of $T_{tt}$. The higher curvature counter terms on the left hand side of the Einstein equation (2.2) are not needed for infinite renormalizations, they will play a rôle, however, in the discussion of the trace anomaly at the end of section 5.

Instead of the pressure we consider the trace of the energy-momentum tensor. We denote it by $\mathcal{T} = \mathcal{E} - (n-1)\mathcal{P}$. For the background field we find

$$\begin{aligned}
\mathcal{T}_{\rm cond} &= n\mathcal{V}_\epsilon(\tilde\phi) \\
&+2(n-1)\left[\xi+\delta\xi-\frac{(n-2)(1+\delta Z)}{4n-4}\right] a^{-2+\epsilon}\left(a^{-1}\tilde\phi' - \frac{n-2}{2}H\tilde\phi\right)^2 \\
&+2(n-1)(\zeta+\delta\zeta)a^{-3+\epsilon/2}\left[\tilde\phi'' + \frac{(n-2)Ra^2}{n-1}\tilde\phi\right] + \frac{\delta Z_{\rm G}}{\kappa}G^\mu_\mu + n\frac{\delta\Lambda}{\kappa}\ ,
\end{aligned}$$

including again counter terms from the left hand side of the Einstein equations.

The fluctuation parts of the energy-momentum tensor are given, after conformal rescaling, by

$$\begin{aligned}
\mathcal{E}_{\rm fl}(\tau) &= a^{-4+\epsilon}\langle\overline{\tilde\psi}\left(\beta\tilde{\mathcal{H}}_p\right)\tilde\psi\rangle \\
&= a^{-4+\epsilon}\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\overline{V}_{-\mathbf{p},s}(\tau)\left(\beta\tilde{\mathcal{H}}_p\right)V_{-\mathbf{p},s}(\tau) \\
&= 2a^{-4+\epsilon}\int\frac{d^{n-1}p}{(2\pi)^{n-1}2E_0}\left\{i\left[E_0-m_0\right)\right]\left(f_p f_p^{*\prime} - f_p' f_p^*\right) - 2E_0 m(\tau)\right\}\ .
\end{aligned} \tag{4.3}$$

for the energy density and by

$$\begin{aligned} \mathcal{P}_{\rm fl}(\tau) &= \frac{a^{-4+\epsilon}}{n-1}\langle\overline{\tilde{\psi}}\boldsymbol{\gamma}\boldsymbol{p}\tilde{\psi}\rangle \\ &= \frac{a^{-4+\epsilon}}{n-1}\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\overline{V}_{-\mathbf{p},s}(\tau)\boldsymbol{\gamma}\boldsymbol{p}V_{-\mathbf{p},s}(\tau) \end{aligned} \tag{4.4}$$

for the pressure. The fluctuation part of the energy-momentum tensor is given by

$$\left(T^\mu_\mu\right)_{\rm fl} = \mathcal{E} - (n-1)\mathcal{P} \; . \tag{4.5}$$

This results in

$$\begin{aligned} \mathcal{T}_{\rm fl}(\tau) &= a^{-4+\epsilon}\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\overline{V}_{-\mathbf{p},s}(\tau)\left(\beta\tilde{\mathcal{H}}_p - \boldsymbol{\gamma}\boldsymbol{p}\right)V_{-\mathbf{p},s}(\tau) \\ &= a^{-4+\epsilon}\sum_s\int\frac{d^3p}{(2\pi)^3 2E_0}\overline{V}_{-\mathbf{p},s}(\tau)m(\tau)V_{-\mathbf{p},s}(\tau) \\ &= m(\tau)a^{-4+\epsilon}\mathcal{F}(\tau) \; . \end{aligned} \tag{4.6}$$

So the trace can be computed by multiplying the fluctuation integral $\mathcal{F}$ by the time-dependent mass.

# 5 Renormalization

In order to develop the framework for renormalizing the one-loop equations, we write the equation of motion for the mode functions, Eq. (3.22), in the form

$$\left[\frac{d^2}{d\tau^2} + E_0^2\right] f_p(\tau) = -V(\tau) f_p(\tau) \; , \tag{5.1}$$

with

$$V(\tau) = m^2(\tau) - m(0)^2 - i\left[m'(\tau) - m'(0)\right] \; . \tag{5.2}$$

Using the initial conditions (3.24) this equation can be recast into the form of an integral equation:

$$f_p(\tau) = e^{-iE_0\tau} - \frac{1}{E_0}\int_0^\tau d\tau' \sin[E_0(\tau-\tau')]V(\tau')f_p(\tau') \; . \tag{5.3}$$

Using this integral equation, the mode functions may be expanded with respect to the potential $V(\tau)$. We split off the zeroth order (plane wave) contribution and an oscillating phase factor by writing

$$f_p(\tau) = e^{-iE_0\tau}[1 + h_p(\tau)] \; . \tag{5.4}$$

The differential and integral equations satisfied by the function $h_p(\tau)$ can easily be derived from Eqs. (5.1) and (5.3), respectively. It may be decomposed as

$$h_p(\tau) = \sum_{n=1}^{\infty} h_p^{(n)}(\tau), \tag{5.5}$$

where $h_p^{(n)}(\tau)$ is of $n$'th order in $V(\tau)$; we define further the inclusive sums

$$h_p^{\overline{(n)}} = \sum_{m=n}^{\infty} h_p^{(m)} \; . \tag{5.6}$$

The truncated mode functions $h_p^{(n)}(\tau)$ and $h_p^{\overline{(n)}}$ satisfy a recursive set of differential and integral equations derived from Eqs. (5.1) and (5.3); these can be used to compute them numerically and analytically. Moreover the expansions (5.5) and (5.6) are expansions with respect to $1/E_0$, we have used them [18] in order to single out the leading divergent contributions in the various mode integrals.

The integrand of the fluctuation integral $\mathcal{F}$, see Eq. (3.26), can be written as

$$\begin{aligned} 1 - \left(1 - \frac{m_0}{E_0}\right)|f_p(\tau)|^2 &= \frac{m_0}{E_0} - \left(1 - \frac{m_0}{E_0}\right)\left[2\mathrm{Re}\, h_p(\tau) + |h_p(\tau)|^2\right] \\ &= \frac{m(\tau)}{E_0} - \frac{m''(\tau)}{4(E_0)^3} - \frac{m^3(\tau)}{2(E_0)^3} + \frac{m(\tau)m_0^2}{2(E_0)^3} \\ &\quad + \frac{m''(0)}{4(E_0)^3}\cos(2E_0\tau) + K_{\mathrm{F}}(p,\tau) \; . \end{aligned} \tag{5.7}$$

The first terms on the right hand side lead to divergent or singular momentum integrals. The function $K_{\mathrm{F}}(\tau)$ can be considered being defined by this equation, i.e. by subtraction of the remainder of the right hand side from the left hand side. It behaves as $(E_0)^{-4}$ and its momentum integral is finite. While other authors [15] use the cutoff independence with scale dependent couplings in order to check their numerical scheme, we use this asymptotic

behavior as a numerical cross check. If defining $K_{\rm F}(\tau)$ by subtraction is numerically precarious, it can also be computed directly, using the reduced mode functions $h_p^{(n)}(\tau)$ and $\overline{h_p^{(n)}}$. This is discussed in [18]. For fermions these expressions get rather lengthy, we do not present them here. Using the truncated mode functions instead of subtracting plainly the divergent parts from the integrand constitutes another numerical and analytical cross-check.

We decompose the fluctuation integral as

$$\mathcal{F}(\tau) = \mathcal{F}_{\rm div}(\tau) + \mathcal{F}_{\rm sing}(\tau) + \mathcal{F}_{\rm fin}(\tau) \; , \tag{5.8}$$

with

$$\begin{aligned}
\mathcal{F}_{\rm div} &= -2\int \frac{d^{n-1}p}{(2\pi)^{n-1}} \left\{ \frac{m(\tau)}{E_0} - \frac{m''(\tau)}{4(E_0)^3} - \frac{m^3(\tau)}{2(E_0)^3} + \frac{m(\tau)m_0^2}{2(E_0)^3} \right\} \; , && (5.9)\\
\mathcal{F}_{\rm sing} &= -2\int \frac{d^3p}{(2\pi)^3} \left\{ -\frac{m'(0)}{2(E_0)^2}\sin(2E_0\tau) + \frac{m''(0)}{4(E_0)^3}\cos(2E_0\tau) \right\} \; , && (5.10)\\
\mathcal{F}_{\rm fin} &= -2\int \frac{d^3p}{(2\pi)^3} K_{\rm F}(p,\tau) \; . && (5.11)
\end{aligned}$$

We have dropped dimensional regularization for the singular and finite parts. They are to be evaluated at $n-4=\epsilon=0$. The divergent part has to be compensated by suitable counter terms. The part $\mathcal{F}_{\rm sing}$ displays a singularity in $\tau$ at $\tau=0$, a phenomenon related to the initial conditions. One gets rid of the problem by a Bogoliubov transformation [33, 34], for fermions its explicit form has been derived in [34, 35]. We derive the form of this Bogoliubov transform for the case under consideration in section 6.

The fluctuation parts of the energy momentum tensor can be analyzed in a similar way. The integrand of the energy density $\mathcal{E}_{\rm fl}$ can be expanded as

$$\begin{aligned}
&\frac{i}{2}\left(1-\frac{m_0}{E_0}\right)\left(f_p f_p^{*\prime} - f_p' f_p^*\right) - m(\tau)\\
&= -(E_0-m_0)\left\{1 + 2{\rm Re}\, h_p + |h_p|^2 - \frac{1}{E_0}{\rm Im}\,\left[h_p'\left(1+h_p^*\right)\right]\right\} - m(\tau)\\
&= -E_0 - \frac{m^2(\tau)}{2E_0} + \frac{m_0^2}{2E_0} + \frac{m'^2(\tau)}{8(E_0)^3} + \frac{m^4(\tau)}{8(E_0)^3}\\
&\quad + \frac{m_0^4}{8(E_0)^3} - \frac{m^2(\tau)m_0^2}{4(E_0)^3} + K_{\rm E}(p,\tau) \; .
\end{aligned} \tag{5.12}$$

Again $K_{\rm E}(p,\tau)$ is defined by this equation and it behaves as $(E_0)^{-4}$ as $E_0 \to \infty$. There is no cosine term here and, therefore, no singular contribution. So

$$\mathcal{E}_{\rm fl}(\tau) = \mathcal{E}_{\rm fl,div}(\tau) + \mathcal{E}_{\rm fl,fin}(\tau) \; , \tag{5.13}$$

with

$$\begin{aligned}
\mathcal{E}_{\rm div} &= 2a^{-4+\epsilon} \int \frac{d^{n-1}p}{(2\pi)^{n-1}} \left\{ -E_0 - \frac{m^2(\tau)}{2E_0} + \frac{m_0^2}{2E_0} + \frac{m'^2(\tau)}{8(E_0)^3} + \frac{m^4(\tau)}{8(E_0)^3} \right. \\
&\qquad \left. + \frac{m_0^4}{8(E_0)^3} - \frac{m^2(\tau)m_0^2}{4(E_0)^3} \right\} \; , & (5.14) \\
\mathcal{E}_{\rm fin} &= 2a^{-4} \int \frac{d^3p}{(2\pi)^3} K_{\rm E}(p,\tau) \; . & (5.15)
\end{aligned}$$

The contribution of the fluctuations to the trace is particularly simple: as explained in the previous section, it is proportionla to the fluctuation integral. So, $\mathcal{T}_{\rm fl}(\tau)$ can be decomposed as

$$\mathcal{T}_{\rm fl}(\tau) = \mathcal{T}_{\rm fl,div}(\tau) + \mathcal{T}_{\rm fl,sing}(\tau) + \mathcal{T}_{\rm fl,fin}(\tau) \; , \tag{5.16}$$

where the three expressions on the right hand side are obtained from those of Eq. (5.8) by multiplying them with $m(\tau)a^{-4+\epsilon}$. For discussing the renormalization we need in particular the divergent part

$$\mathcal{T}_{\rm fl,div} = a^{-4+\epsilon} \int \frac{d^{n-1}p}{(2\pi)^{n-1}} \left\{ \frac{m^2(\tau)}{E_0} - \frac{m(\tau)m''(\tau)}{4(E_0)^3} - \frac{m^4(\tau)}{2(E_0)^3} + \frac{m^2(\tau)m_0^2}{2(E_0)^3} \right\} \tag{5.17}$$

The divergent terms $\mathcal{F}_{\rm div}, \mathcal{E}_{\rm div}$, and $\mathcal{T}_{\rm div}$ are proportional to local terms in $\phi(\tau)$ and its derivatives. These can be absorbed in the usual way by introducing the appropriate counter terms into the Lagrangian and into the energy-momentum tensor.

The divergent parts of the fluctuation integral can be evaluated, e.g., using dimensional regularization. One finds

$$\mathcal{F}_{\rm div} = 2m''(\tau)L_0 + 4m^3(\tau)L_0 + \frac{m(\tau)m_0^2}{4\pi^2} \; , \tag{5.18}$$

with the abbreviation [3]

$$L_0 = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2 a^2(\tau)}{m_0^2} - \gamma \; . \right\} \tag{5.19}$$

[3] This definition and the one of $L$ and $L_f$ below deviate from those in our previous work [18, 10] by inclusion of the factor $1/16\pi^2$.

The scale $\mu$ is introduced as usual by 'adjusting the dimension' of the various couplings, we have not displayed this rescaling explicitly. The initial mass only appears in the logarithm of $L_0$ and in the last term of (5.18) which is finite. Therefore, as already found for the scalar fluctuations [11], the dependence on the initial mass $m_0 = m(0)$ can be absorbed into finite terms, $\Delta Z, \Delta\sigma, \ldots$. In order to appreciate this independence of the initial mass it should be noticed that the individual divergent terms appear with prefactors up to $m_0^3$, and up to $m_0^4$ in the energy momentum tensor. If one uses a three-dimensional 'computer' cutoff the necessary cancellations do not occur.

We expand (5.18) using the explicit expression (3.14) for $m(\tau)$. This leads to rather lengthy expressions, in particular we have to keep terms proportional to $n-4$, as these become finite terms when multiplied with the $1/\epsilon$ of $L_0$. A further source of finite terms is the product $L_0 a^{\epsilon/2}$ which appears when $g a^{\epsilon/2}\mathcal{F}$ is evaluated in the equation of motion. We have used MAPLE for computing the divergent terms and their finite remnants in the equation of motion. The second derivative term contains terms proportional to $a''$, and therefore to the curvature scalar $R$. So the conformal couplings $\zeta$ and $\xi$ are needed already in order to renormalize the equation of motion. In flat Minkowski space these couplings only appear in the pressure.

Applying an $\overline{MS}$ prescription, the infinite renormalizations become

$$\begin{aligned} \delta Z &= -2g^2 L & \delta\sigma &= -4g m_f^3 L \\ \delta M^2 &= -12 m_f^2 g^2 L & \delta\kappa &= -24 m_f g^3 \\ \delta\lambda &= -24 g^4 L & \delta\zeta &= -\frac{1}{3} m_f L \\ \delta\xi &= -\tfrac{1}{3} g^2 L & & \end{aligned} \tag{5.20}$$

with

$$L = \frac{1}{16\pi^2}\left\{\frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{M^2} - \gamma\,.\right\} \tag{5.21}$$

We use a scale factor $M$ instead of the fermion mass, as one may want to consider the case $m_f = 0$. The renormalization of the conformal coupling $\delta\xi$ is due entirely to the appearance of $\delta Z$ in the conformal coupling term.

The renormalized equation of motion is obtained by replacing the infinite parts of the counter terms by their finite remnants, and by including some additional terms. The finite remnants of the counter terms are

$$\Delta Z = 2g^2 L_f \qquad \Delta\sigma = 4g m_f^3 L_f$$

$$\begin{aligned} \Delta M^2 = 12m_f^2 g^2 L_f \qquad & \Delta\kappa = 24 m_f g^3 L_f \\ \Delta\lambda = 24 g^4 L_f \qquad & \Delta\zeta = \frac{1}{3} g m_f L_f \\ \Delta\xi = \frac{1}{3} g^2 L_f & \end{aligned} \tag{5.22}$$

with

$$L_f = \frac{1}{16\pi^2} \ln \frac{M^2 a^2(\tau)}{m_0^2} \; . \tag{5.23}$$

These finite parts are *time-dependent*, the term $\ln a(\tau)$ arises from the conformal scale factors via the product $(2/\epsilon)a^{\epsilon/2}$. Conceptually they are part of the remaining nonlocal finite fluctuation integrals, though they can be written in local form.

The renormalized equation of motion, omitting the scalar back-reaction, reads

$$\begin{aligned} &(1+\Delta Z)\tilde{\phi}'' + (\zeta + \Delta\zeta) a^3 R + a^2 \left[\xi - \frac{1}{6}\right] R\tilde{\phi} + a^4 \frac{d\mathcal{V}_{\rm ren}(\tilde{\phi})}{d\tilde{\phi}} \\ &+ \frac{g}{16\pi^2}\left(\frac{2}{9}a^2 m R + 4m_0^2 m + 4aHm' - 2a^2H^2 m\right) \\ &+ g\left(\mathcal{F}_{\rm sing}(\tau) + \mathcal{F}_{\rm fin}(\tau)\right) = 0 \; , \end{aligned} \tag{5.24}$$

where we have introduced the renormalized potential

$$\begin{aligned} \mathcal{V}_{\rm ren}(\tilde{\phi}) \;\; = \;\; & (\sigma + \Delta\sigma) a^{-1}\tilde{\phi} + \frac{M^2 + \Delta M^2}{2} a^{-2}\tilde{\phi}^2 \\ & + \frac{\kappa + \Delta\kappa}{6} a^{-3}\tilde{\phi}^3 + \frac{\lambda + \Delta\lambda}{24} a^{-4}\tilde{\phi}^4 \end{aligned} \tag{5.25}$$

that we will find again in the renormalized stress-energy tensor. We note that $\Delta\xi$ and $\Delta Z$ have cancelled in the conformal coupling term; however, even if $\xi = 1/6$ an additional conformal coupling $\propto g^2$ is introduced by the first term in the parenthesis in Eq. (5.24). It does not vanish even for $m_f = 0$. The cosmological constant counter term has not yet been fixed as it does not appear in the derivative of the potential. It will be determined below. While the equation of motion is now finite for $\tau \neq 0$ it still contains $\mathcal{F}_{\rm sing}(\tau)$ that is singular at $\tau = 0$.

The divergent parts of the energy density give, after dimensional regularization

$$\mathcal{E}_{\rm fl,div} = a^{-4+\epsilon}\left\{m'^2(\tau)L_0 + m^4(\tau)L_0 + \frac{m^4(0)}{32\pi^2} + \frac{m^2(\tau)m_0^2}{8\pi^2}\right\} \; . \tag{5.26}$$

We again expand this, using the explicit expression for $m(\tau)$. In addition to the divergent terms found in the equation of motion we find terms that have to be compensated by the cosmological constant counter term and by the wave function renormalization of the gravitation field. This fixes these couplings to

$$\begin{aligned} \delta\Lambda &= -\kappa m_f^4 L & (5.27) \\ \delta Z_G &= \kappa\frac{m_f^2}{3}L \; , & (5.28) \end{aligned}$$

again independent of the initial condition. The renormalized energy density is given by

$$\begin{aligned} \mathcal{E}_{\rm ren} = \; & \frac{1+\Delta Z}{2}a^{-4}(\tilde{\phi}')^2 + \mathcal{V}_\epsilon(\tilde{\phi}) + 6(\zeta+\Delta\zeta)a^{-2}H\tilde{\phi}' \\ & +6\left[\xi - \frac{1}{6}\right]a^{-2}\left(Ha^{-1}\tilde{\phi}\tilde{\phi}' - \frac{1}{2}H^2\tilde{\phi}^2\right) \\ & +\frac{a^{-4}}{96\pi^2}\left(3m_0^4 + 12m_0^2m^2 + 8aHmm' + 2a^2H^2m^2\right) \\ & +\frac{\Delta Z_G}{\kappa}G_{tt} + \frac{\Delta\Lambda}{\kappa} + \mathcal{E}_{\rm fin} \; , \end{aligned} \tag{5.29}$$

where we have introduced the finite renormalizations

$$\begin{aligned} \Delta\Lambda &= \kappa m_f^4 L_f & (5.30) \\ \Delta Z_G &= -\kappa\frac{m_f^2}{3}L_f \; . & (5.31) \end{aligned}$$

Though these terms have a local form they need not be considered as renormalizations of $Z_G$ and of the cosmological constant, they are just finite parts of the mode integral.

The divergent part of the trace $\mathcal{T} = T_\mu^\mu$ is obtained from the one of $\mathcal{F}_{\rm div}$ by multiplying with $m(\tau)a^{-4+\epsilon}$ as

$$\mathcal{T}_{\rm fl,div} = a^{-4+\epsilon}\left[4m^4L_0 + \frac{m_0^2m^2}{4\pi^2} + 2mm''L_0\right] \; . \tag{5.32}$$

Using the same procedure as above, and inserting all the divergent counter terms as they have already been determined we find the renormalized trace

$$
\begin{aligned}
\mathcal{T}_{\rm ren} \;=\;& 4\mathcal{V}_{\rm ren}(\tilde{\phi}) \\
&+6\left(\xi-\frac{1}{6}\right)a^{-2}\left(a^{-1}\tilde{\phi}'-H\tilde{\phi}\right)^2 \\
&+6(\zeta+\Delta\zeta)a^{-3}\left(\tilde{\phi}''+\frac{2}{3}Ra^{-2}\tilde{\phi}\right)+\frac{\Delta Z_{\rm G}}{\kappa}G^\mu_\mu+4\frac{\Delta\Lambda}{\kappa} \\
&+\frac{a^{-4}}{48\pi^2}\left(a^2Rm^2-8a^2H^2m^2+12m_0^2m^2+4mm''\right. \\
&\left.+6m^4-2m'^2+16aHmm'\right) \\
&+m\left(\mathcal{F}_{\rm sing}+\mathcal{F}_{\rm fin}\right)\ .
\end{aligned}
\tag{5.33}
$$

Thus far we have discussed the way of handling the infinite terms by renormalization of the bare parameters. We also have considered carefully the finite terms left over after renormalization, in particular those that appear due to the continuation of the space-time dimension via divergent factors $1/(n-4)$ multiplying terms that vanish as $n-4$ as $n\to 4$. We have considered, however, only a restricted class of metrics, conformally flat ones. Furthermore, while our regularization is relativistically covariant it is not so under general coordinate transformations. As a consequence, the anomalous contributions do not appear (they neither do, for the same reasons, in Ref. [31] where a Pauli-Villars regularization is used). Indeed for the Dirac field we seemingly need no counter terms of the form ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$ or $H_{\mu\nu}$. In general, however, one expects the divergent part of $T_{\mu\nu}$ for non-interacting fermions to be (see, e.g., [28, 29, 30])

$$
T^{\rm div}_{\mu\nu}=\left\{\frac{8m^4}{n(n-2)}g_{\mu\nu}-\frac{2m^2}{3(n-2)}G_{\mu\nu}-\frac{1}{180}\left[\frac{7}{4}H_{\mu\nu}+2\,{}^{(2)}H_{\mu\nu}-\frac{5}{4}\,{}^{(1)}H_{\mu\nu}\right]\right\}L\ , \tag{5.34}
$$

where $L$ has been defined in the previous section, Eq. (5.21). The first two terms in the parenthesis are cancelled by the cosmological constant and gravitational wave function renormalization counter terms, Eq. (5.27), they agree with the divergences we find in our analysis. The remaining terms are higher curvature terms, for whom we have not found any divergences. This is due to the fact that for conformally flat metrics in $n=4$ one has the identity

$$
H_{\mu\nu}={}^{(2)}H_{\mu\nu}=\frac{1}{3}\,{}^{(1)}H_{\mu\nu} \tag{5.35}
$$

and the expression in square brackets, i.e., the entire higher curvature contribution vanishes, in agreement with our analysis of divergent terms. However, we have to supply counter terms for those contributions as well, as for metrics that are not conformally flat such counter terms would be required. That means we have to subtract these terms. Even in $n = 4$ they yield a finite contribution, the anomaly, if one continues the expressions for those tensors from $n \neq 4$ to $n = 4$ and takes into account the factor $2/\epsilon = 2/(n-4)$ in the divergent factor $L$. The explicit expressions for these tensors for $n \neq 4$ are given in Appendix A. The finite contributions to $T_{\mu\nu}$ which we call $T_{\mu\nu}^{\rm an}$ have been evaluated on the basis of these formulae as well as (5.34), using MAPLE. We find

$$\begin{aligned} T_{00}^{\rm an} &= \frac{1}{960\pi^2}\left\{\frac{1}{12}R^2 - H^2 R - \frac{11}{2}H^4 - H\dot{R}\right\} && (5.36) \\ T_{\mu}^{\mu \rm an} &= \frac{1}{960\pi^2}\left\{-3H\dot{R} - \frac{11}{3}H^2 R + 22H^4 - \ddot{R}\right\} \; . && (5.37) \end{aligned}$$

The anomalous part of the energy-momentum tensor is conserved separately from the nonanomalous part.

# 6 The initial singularity

The fluctuation integral, the energy density, and the trace of the energy-momentum tensor contain contributions $\mathcal{F}_{\rm sing}, \mathcal{E}_{\rm sing}$ and $\mathcal{T}_{\rm sing}$ that become singular as $\tau \to 0$. We have discussed previously [18] how to get rid of such singularities by a Bogoliubov transformation. There we had assumed that $\dot{m}(0) = 0$. Here $m'(0)$ is necessarily different from zero, as $a'(0) = H(0)a(0)^2$ is different from zero due to the first Friedmann equation. So the singular terms are different and the singularities are stronger. For the Minkowski geometry this has been discussed in [35]. From Eq. (5.10) we have, using explicit expressions given in Appendix B of Ref. [34],

$$\mathcal{F}_{\rm sing}(\tau) \simeq \frac{m'(0)}{4\pi^2}\tau^{-1} + \frac{m''(0)}{4\pi^2}\ln\tau \; . \qquad (6.1)$$

The energy density stays finite as $\tau \to 0$ and the trace behaves as

$$\mathcal{T}_{\rm sing}(\tau) \simeq m(\tau)\left\{\frac{m'(0)}{4\pi^2}\tau^{-1}\ln\tau + \frac{m''(0)}{4\pi^2}\ln\tau\right\} \; . \qquad (6.2)$$

As it is simply proportional to the fluctuation integral, it is sufficient to render the fluctuation integral finite. To do so one performs a Bogoliubov transformation

$$
\begin{aligned}
b_{\mathbf{p},s} &= \cos(\beta_{\mathbf{p},s})\tilde{b}_{\mathbf{p},s} + \sin(\beta_{\mathbf{p},s}) \exp^{i\delta_{\mathbf{p},s}} \tilde{d}^{\dagger}_{-\mathbf{p},s} \qquad (6.3)\\
d^{\dagger}_{-\mathbf{p},s} &= -\sin(\beta_{\mathbf{p},s})e^{-i\delta_{\mathbf{p},s}}\tilde{b}_{\mathbf{p},s} + \cos(\beta_{\mathbf{p},s})\tilde{d}^{\dagger}_{-\mathbf{p},s} \qquad (6.4)
\end{aligned}
$$

and defines the modified initial state as being annihilated by the new operators $\tilde{b}_{\mathbf{p},s}$ and $\tilde{d}^{\dagger}_{-\mathbf{p},s}$. The fluctuation integral then takes the form

$$
\begin{aligned}
\tilde{\mathcal{F}}(\tau) = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_0} \Big\{ & \bar{U}_{\mathbf{p},s}U_{\mathbf{p},s}\sin^2\beta_{\mathbf{p},s} + \bar{V}_{-\mathbf{p},s}V_{\mathbf{p},s}\cos^2\beta_{\mathbf{p},s} \\
& + \bar{V}_{-\mathbf{p},s}U_{\mathbf{p},s}e^{i\delta_{\mathbf{p},s}}\cos\beta_{\mathbf{p},s}\sin\beta_{\mathbf{p},s} \qquad (6.5)\\
& + \bar{U}_{\mathbf{p},s}V_{-\mathbf{p},s}e^{-i\delta_{\mathbf{p},s}}\cos\beta_{\mathbf{p},s}\sin\beta_{\mathbf{p},s}\Big\} \ .
\end{aligned}
$$

or, explicitly

$$
\begin{aligned}
\tilde{\mathcal{F}}(\tau) = -2\sum_s \int \frac{d^{n-1}p}{(2\pi)^{n-1}} \Bigg\{ & \cos(2\beta_{\mathbf{p},s}) \left[ \frac{m(\tau)}{E_0} - \frac{m''(\tau)}{4(E_0)^3} - \frac{m^3(\tau)}{2(E_0)^3} + \frac{m(\tau)m_0^2}{2(E_0)^3} \right.\\
& \left. - \frac{m'(0)}{2(E_0)^2}\sin(2E_0\tau) + \frac{m''(0)}{4(E_0)^3}\cos(2E_0\tau) + K_F(p,\tau) \right] \qquad (6.6)\\
& + \sin(2\beta_{\mathbf{p},s})\frac{ps}{2E_0(E_0+m_0)}\left[-2\mathrm{Im}\left(f_p f_p' e^{i\delta_{\mathbf{p},s}}\right) + 2m\mathrm{Re}\left(f_p^2 e^{i\delta_{\mathbf{p},s}}\right)\right]\Bigg\}
\end{aligned}
$$

Here we have used Eq. (5.7) and $\bar{U}_{\mathbf{p},s}U_{\mathbf{p},s} = -\bar{V}_{\mathbf{p},s}V_{\mathbf{p},s}$ as well as

$$
\bar{V}_{-\mathbf{p},s}U_{\mathbf{p},s} = \left(\bar{U}_{\mathbf{p},s}V_{-\mathbf{p},s}\right)^* = \frac{2ps}{E_0+m_0}\left(if_p f_p' + m(\tau)f_p^2\right) \ . \qquad (6.7)
$$

In determining the parameters $\beta_{\mathbf{p},s}$ and $\delta_{\mathbf{p},s}$ the mode functions in the last bracket of Eq (6.6) can be replaced by their leading behavior

$$
\begin{aligned}
2\mathrm{Im}\left(f_p f_p' e^{i\delta_{\mathbf{p},s}}\right) &\simeq 2E_0\cos(2E_0\tau - \delta_{\mathbf{p},s}) \qquad (6.8)\\
2m(\tau)\mathrm{Re}\left(f_p^2 e^{i\delta_{\mathbf{p},s}}\right) &\simeq 2m(\tau)\cos(2E_0\tau - \delta_{\mathbf{p},s}) \ . \qquad (6.9)
\end{aligned}
$$

As we discuss the limiting behavior as $\tau \to 0$ we can replace $m(\tau)$ by $m_0$. We then obtain the following conditions for the cancellation of the singular

contributions

$$-\cos(2\beta_{\mathbf{p},s})\frac{m''(0)}{2E_0^2}+2ps\sin 2\beta_{\mathbf{p},s}\cos(\delta_{\mathbf{p},s})=0 \qquad (6.10)$$

$$\cos(2\beta_{\mathbf{p},s})\frac{m'(0)}{E_0}+2ps\sin 2\beta_{\mathbf{p},s}\sin(\delta_{\mathbf{p},s})=0 \qquad (6.11)$$

so that the parameters are obtained as [4]

$$\tan\delta_{\mathbf{p},s} = -2E_0\frac{m'(0)}{m''(0)} \qquad (6.12)$$

$$\tan 2\beta_{\mathbf{p},s} = \frac{\ddot{m}(0)}{4sp(E_0)^2}\sqrt{1+\tan^2\delta}\ . \qquad (6.13)$$

Here we have retained only the leading asymptotic behavior. The resulting modifications of the various mode function integrals can be read off from those of Ref. [18]. One should note that $m''(0)$ does depend, via the equation of motion for $\phi(\tau)$, on the modified fluctuation integral $\mathcal{F}(0)$ which now does not vanish as $\tau\to 0$, and this again depends on $m''(0)$. So $m''(0)$ has to be determined self-consistently.

The situation *before* removing the initial singularity is actually even worse than it is apparent in the analysis of $\mathcal{F}_{\rm sing}$. By the second Friedmann equation a singularity of $\mathcal{T}(\tau)$ entails a singular behaviour of $a''(\tau)$, but $m(\tau)$ contains a contribution $a''(\tau)m_f$, so $m''(0)$ does not even exist. This just reflects the impossibility of starting the evolution consistently without getting rid, beforehand, of the initial singularity. An alternative way of avoiding these initial singularities has been discussed recently for scalar fields, using adiabatic subtraction [15].

# 7 Conclusions

We have derived here the renormalized equations of motion for a massive Dirac field, with scalar coupling to a Yukawa field, in a flat FRW universe. We have used a formalism - essentially a resolvent expansion - that is useful for numerical computations and allows at the same time for a straightforward

[4]The right hand side of Eq. (V.9) in [18] should have a factor 4 in the denominator, instead of 8, in accordance with Eq. (V.10).

analysis of the divergent terms and their separation from the finite parts to be computed numerically. Straightforward here does not mean simple. For a massive Dirac field the Feynman diagrams for the effective action are divergent up to diagrams with 4 external scalar fields. This reflects itself in a rather lengthy analysis, in our formalism as well. We have not reproduced all the details here, but rely on our previous work [18] for a Yukawa theory in Minkowski space.

The divergent terms obtained in our formalism have been found in consistency with standard results, and the renormalization counter terms can be chosen independent of the initial conditions. The fact that we have considered a massive fermion field leads to a larger number of couplings of the scalar field, necessary for renormalization. As these may be obtained by a shift of the scalar field, this modification is a straightforward generalization. The renormalization counter terms have been determined in dimensional regularization and with a $\overline{MS}$ convention for the renormalization. The formalism can be adapted to other regularization and renormalization schemes.

The trace anomaly does not appear explicitly in our formalism. This is a consequence of the special geometry we consider here and does not imply an inconsistency of the formalism. This can be understood from the general analysis of the divergent parts of the energy-momentum tensor, as given, e.g., in [30]. We have used this analysis in order to infer the anomalous part of the energy-momentum tensor.

We have also analyzed the initial singularities. They obstruct the cosmological initial value problem in an essential way, a singularity in the trace of the energy momentum tensor leads to a singularity in the expansion rate. So they have to be avoided by a suitable choice of the initial state. We have constructed explicitly such a state; it is not unique but can be modified if only the asymptotic behavior as $p \to \infty$ of the Bogoliubov parameters is retained.

Our results can be used in a straightforward way for numerical computations, which we plan to carry out in the near future. We do not expect an essential modification of results related to the parametric resonance phenomenon; however, especially for large Yukawa couplings and/or masses, the finite remnants of the renormalization counter terms and the singular parts are not necessarily small and this signals the importance of considering renormalization carefully.

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## A The higher curvature tensors

The higher curvature tensors are defined by

$$\begin{aligned}
{}^{(1)}H_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^n x\sqrt{-g}\,R^2 \\
&= 2\,R_{;\mu\nu} - 2g_{\mu\nu}\,\Box R - \frac{1}{2}g_{\mu\nu}\,R^2 + 2RR_{\mu\nu}\ , \qquad \text{(A.1)}\\
{}^{(2)}H_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^n x\sqrt{-g}\,R^{\alpha\beta}R_{\alpha\beta} \\
&= 2R^{\ \alpha}_{\mu\ ;\nu\alpha} - \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,\Box R + 2R^{\ \alpha}_{\mu}R_{\alpha\nu} - \frac{1}{2}g_{\mu\nu}\,R^{\alpha\beta}R_{\alpha\beta} \\
H_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^n x\sqrt{-g}\,R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \\
&= -\frac{1}{2}g_{\mu\nu}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + 2R^{\mu\alpha\beta\nu}R^{\ \alpha\beta\gamma}_{\nu} - 4\Box R_{\mu\nu} + 2R_{;\mu\nu} \\
&\quad -4R_{\mu\alpha}R^{\ \alpha}_{\nu} + 4R^{\alpha\beta}R_{\alpha\mu\beta\nu}\ . \qquad \text{(A.2)}
\end{aligned}$$

On order to evaluate the anomalous contribution to the energy-momentum tensor we need the continuation of the higher order curvature tensors for the FRW geometry to $n = 4 - \epsilon$ dimensions. We choose to continue the spatial dimension to $3 - \epsilon$. We already have given the $n$-dimensional continuation of the Ricci tensor in Eq. (2.7).

For the time-time components and the trace of the tensors ${}^{(n)}H_{\mu\nu}$ we obtain

$$ {}^{(1)}H_{tt} = -6H\dot{R} + \frac{1}{2}R^2 - 6H^2 R \qquad \text{(A.3)}$$

$$
\begin{aligned}
& +(n-4)\left(-2H\dot{R}-(n+1)RH^2\right) , \\
{}^{(2)}H_{\ tt} &= -2H\dot{R}+\frac{1}{6}R^2-2H^2R+(n-4)\left(-\frac{1}{2}H\dot{R}\right. \qquad \text{(A.4)} \\
& \left. -\frac{R^2}{24(n-1)}-\frac{1}{4}(n+2)H^2R+\frac{1}{8}(n-1)(n-2)^2H^4\right) , \\
H_{\ tt} &= -2H\dot{R}+\frac{1}{6}R^2-2H^2R \qquad \text{(A.5)} \\
& +(n-4)\left(-\frac{R^2}{6(n-1)}-H^2R+\frac{1}{2}(n-1)(n-2)H^4\right) ,
\end{aligned}
$$

$$
\begin{aligned}
{}^{(1)}H^{\ \mu}_{\ \mu} &= -6\ddot{R}-18H\dot{R}+(n-4)\left(-2\ddot{R}-2(n+2)H\dot{R}-\frac{1}{2}R^2\right) , \qquad \text{(A.6)} \\
{}^{(2)}H^{\ \mu}_{\ \mu} &= -2\ddot{R}-6H\dot{R}+(n-4)\left(-\frac{1}{2}\ddot{R}-\frac{1}{2}(n+3)H\dot{R}\right. \qquad \text{(A.7)} \\
& \left. -\frac{nR^2}{8(n-1)}+\frac{1}{4}(n-2)^2H^2R-\frac{1}{8}n(n-1)(n-2)^2H^4\right) , \\
H^{\ \mu}_{\ \mu} &= -2\ddot{R}-6H\dot{R}+(n-4)\left(-2H\dot{R}-\frac{R^2}{2(n-1)}\right. \\
& \left. +(n-2)H^2R-\frac{1}{2}n(n-1)(n-2)H^4\right) . \qquad \text{(A.8)}
\end{aligned}
$$

# B The Dirac equation in $n$ dimensional FRW geometry

The Dirac equation in curved space is formulated using the $n$-bein $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^{ab} = \omega^{ab}_\nu dx^\nu$. It has the form

$$
i\gamma^a E^\nu_a \left(\partial_\nu + \omega^{ab}_\nu \frac{1}{8}\left[\gamma_a, \gamma_b\right]\right)\psi - m\psi = 0 \ . \qquad \text{(B.1)}
$$

Here $E^\mu_a$ is the inverse $n$-Bein and the connection is defined by the equation

$$
d \wedge e^a + \omega^a_b \wedge e^b = 0 \qquad \text{(B.2)}
$$

Choosing for the FRW geometry $e^0 = dt$ and $e^k = a\, dx^k$ one finds

$$
\omega^k{}_0 = \dot{a}\, dx^k \ , \qquad \text{(B.3)}
$$

while the purely spatial components $\omega^k{}_j$ vanish. Using $E^0_0 = 1$, $E^k_k = 1/a$, all other components being zero, we obtain

$$i\left(\gamma^0\partial_0 + \frac{1}{a}\gamma^k\nabla_k + \frac{1}{a}\gamma^k\omega^{k0}{}_k\frac{1}{4}\left[\gamma_k,\gamma_0\right]\right)\psi - m\psi = 0\ , \tag{B.4}$$

or, using $\gamma^k\left[\gamma_k,\gamma_0\right] = 2(n-1)\gamma^0$

$$i\gamma^0\left(\partial_0 + \frac{n-1}{2}\frac{\dot{a}}{a}\right)\psi + \left(\frac{i}{a}\gamma^k\nabla_k - m\right)\psi = 0\ . \tag{B.5}$$

It is the almost obvious generalization of the Dirac equation in the $n = 4$ FRW geometry.

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